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ON VOIDS OF MINIMUM STRESS CONCENTRATION

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Abstract—An isotropic elastic medium containing a void is loaded at infinity by given stresses. The problem of finding a minimizing void shape for the stress concentration is formulated. It is proved that a sufficient condition for a surface to be a minimizer is that the two surface principal stresses be constant and equal. A class of ellipsoids having this property is exhibited and relations between the applied stresses and the ellipsoid parameters are established.

1. INTRODUCTION

In this paper, we formulate and study the problem of determining a cavity shape which minimizes the stress concentration, if the cavity lies within an infinite homogeneous and isotropic elastic medium, and the stress at infinity is given. The stress concentration is minimized by minimizing the maximum stress in the euclidean norm.

In Section 2, the variational problem is formulated, and in Section 3 a sufficient condition for a shape to be a minimizer is established. This condition, which stipulates that the two non-vanishing principal stresses be constant and equal at the void surface, is an extension tc three-dimensional states of stress of the "constant-stress surface" condition studied in[1, 2] in connection with the more elementary cases of axisymmetric torsion and plane deformation.

In Section 4, we show that surfaces with the property arrived at in Section 3 can be realized as ellipsoids, provided the applied stress is suitably restricted. Using results from[3], we arrive at relations between the components of the applied stress and the parameters of the ellipsoid.

2. FORMULATION OF THE PROBLEM

Consider an infinite homogeneous and isotropic elastic medium with Lamé moduli λ, μ containing a cavity with surface Ω. Let V be the region bounded externally by Ω and let V^- denote its complement. We assume that Ω is suitably regular.

We suppose that body force is absent and assume that the stress $\sigma(x)$ in V^- satisfies the usual elasticity equations, the free surface conditions on Ω and the condition at infinity

$$\sigma(x) = \sigma^\circ + o(1) \quad \text{as } |x| \to \infty, \tag{2.1}$$

where σ° is a constant tensor. It is important in what follows that the last condition is equivalent to (see [4, 5])

$$\sigma(x) = \sigma^\circ + o(|x|^{-1}) \quad \text{as } |x| \to \infty. \tag{2.2}$$

Let us introduce the norm $\|\sigma\|$ of $\sigma(x)$ defined by

$$\|\sigma\| = V^-|\sigma|, \quad \text{where } |\sigma| = \surd(\mathrm{tr}\, \sigma^2). \tag{2.3}$$

It is convenient to extend $\sigma(x)$ as zero inside Ω and introduce the function space $\mathscr{F}(\lambda, \mu, \sigma^\circ)$ consisting of all σ having bounded norm.

We say that Ω_* is a minimizing surface if the corresponding stress field σ^* is such that

$$\|\sigma\| \geq \|\sigma^*\| \quad \text{for every } \sigma \in \mathscr{F}(\lambda, \mu, \sigma^\circ). \tag{2.4}$$

The general variational problem is to describe the set of all minimizing surfaces Ω_*. In this paper we find a class of minimizing surfaces by suitably restricting the stress at infinity σ°.

3. SURFACES OF CONSTANT AND EQUAL PRINCIPAL STRESSES

At a free surface there are at most two non-zero principal stresses. Here we consider surfaces such that these stresses are both equal and constant over the whole surface. We show that such surfaces are minimizers.

Theorem. Let $\sigma^* \in \mathcal{F}(\lambda, \mu, \sigma^\circ)$ be such that

$$\sigma^*_{ij} = \tau(\delta_{ij} - n_i n_j) \text{ on } \Omega_*, \tag{3.1}$$

where τ is a constant and n_i are the components of the unit normal to Ω_*. Then:

(a) Ω_* is a minimizing surface;

$$\text{(b) } \|\sigma^*\| = \max\left\{\frac{1}{\sqrt{2}}|\text{tr}\,\sigma^\circ|, |\sigma^\circ|\right\}; \tag{3.2}$$

$$\text{(c) } \tau = \frac{1}{2}\text{tr}\,\sigma^\circ. \tag{3.3}$$

In order to prove this theorem, we need the following result from potential theory.

Lemma. Let ϕ be continuous on Ω and satisfy

$$\Delta\phi = 0 \text{ on } V^-, \qquad \phi = a + o(|x|^{-1}) \quad \text{as } |x| \to \infty, \tag{3.4}$$

where a is a constant. Then:

(a) either $|\phi|$ exceeds $|a|$ at a point of Ω or $\phi = a$ on V^-;

(b) if ϕ is constant on Ω, then $\phi = a$ on V^-.

One can establish this lemma by transforming the difference $\phi - a$ through a Kelvin inversion to a function which is harmonic on a bounded region and applying the maximum principle (see, e.g. [6]).

Proof of the Theorem. First we will prove (3.3) and (3.2). As is well known,

$$\Delta\,\text{tr}\,\sigma^* = 0 \text{ on } V_*^-. \tag{3.5}$$

From (3.1), we get $\text{tr}\,\sigma^* = 2\tau$ on Ω_* and from (2.2) there follows

$$\text{tr}\,\sigma^* = \text{tr}\,\sigma^\circ + o(|x|^{-1}) \quad \text{as } |x| \to \infty. \tag{3.6}$$

Thus, Part (b) of the lemma furnishes the conclusion that $\text{tr}\,\sigma^*$ is constant on V_*^-,

$$\text{tr}\,\sigma^* = \text{tr}\,\sigma^\circ = 2\tau \text{ on } V_*^-, \tag{3.7}$$

which disposes of (3.3).

Using the fact that $\text{tr}\,\sigma^* = \text{const.}$, the field equations yield

$$\Delta\sigma^* = 0 \text{ on } V_*^-. \tag{3.8}$$

Accordingly,

$$\Delta|\sigma|^2 = 2\sigma^*_{ij,k}\sigma^*_{ij,k} \geq \text{ on } V_*^-, \tag{3.9}$$

which shows that $|\sigma^*|^2$ is subharmonic on V_*^- and therefore obeys the maximum principle.

Thus,

$$\|\sigma^*\| = \max\{\sup_{\Omega_*}|\sigma^*|, |\sigma^\circ|\}. \tag{3.10}$$

From (3.1), (3.7), there follows

$$\sup_{\Omega^*}|\sigma^*| = \frac{1}{\sqrt{2}}|\mathrm{tr}\,\sigma^\circ|, \tag{3.11}$$

which concludes the proof of (3.2).

It remains to be shown that if $\sigma \in \mathscr{F}(\lambda, \mu, \sigma^\circ)$, then $\|\sigma\| \geq \|\sigma^*\|$. We have two cases to consider. From (3.2), we know that either

$$\|\sigma^*\| = |\sigma^\circ|, \quad \text{or } \|\sigma^*\| = \frac{1}{\sqrt{2}}|\mathrm{tr}\,\sigma^\circ|. \tag{3.12}$$

The first case is trivial because $\|\sigma\| \geq |\sigma^\circ|$.

Assume the second of (3.12) holds. Clearly, $\|\sigma\| \geq \sup_{\Omega}|\sigma|$. It is easy to show that for a tensor σ with one principal stress zero, $\sqrt{2}|\sigma| \geq |\mathrm{tr}\,\sigma|$. Thus,

$$\|\sigma\| \geq \frac{1}{\sqrt{2}}\sup_{\Omega}|\mathrm{tr}\,\sigma|. \tag{3.13}$$

Since $\Delta\,\mathrm{tr}\,\sigma = 0$, and in view of (2.2), Part (a) of the lemma implies

$$\sup_{\Omega}|\mathrm{tr}\,\sigma| \geq |\mathrm{tr}\,\sigma^\circ|, \tag{3.14}$$

which together with (3.13) furnishes

$$\|\sigma\| \geq \frac{1}{\sqrt{2}}|\mathrm{tr}\,\sigma^\circ| = \|\sigma^*\|. \tag{3.15}$$

The proof is now complete.

It is important to note that as seen from (3.1), (3.3), of the two possible values for $\|\sigma^*\|$ indicated in (3.2), the one $\|\sigma^*\| = (1/\sqrt{2})|\mathrm{tr}\,\sigma^\circ|$ is assumed when $|\sigma^*|$ has its maximum at the void surface, whereas the other value corresponds to the case when the maximum is at infinity. Thus, if the applied stess σ° obeys the condition $|\mathrm{tr}\,\sigma^\circ| \geq \sqrt{2}|\sigma^\circ|$, then the maximum value of $|\sigma^*|$ occurs at a point of the void surface. We will return to this point in the next section.

4. ELLIPSOIDS AS MINIMIZERS

Now we will show that surfaces satisfying the conditions of the theorem can be realized as ellipsoids. We will use an explicit expression for the stress at points on the surface of an ellipsoidal void, which was obtained in the work [3].

Let the ellipsoidal void be given by the equation

$$xa^{-2}x = x_i(a^{-2})_{ij}x_j = 1 \tag{4.1}$$

where the tensor a_{ij} has principal values a_1, a_2, a_3 which are the semi-axes of the ellipsoid.

Denote by $n(x)$ the unit normal to the ellipsoid at the point x. The formulas

$$n = \frac{a^{-2}x}{\sqrt{(xa^{-4}x)}}, \quad x = \frac{a^2 n}{\sqrt{(na^2 n)}} \tag{4.2}$$

establish a one-to-one correspondence between points x of the ellipsoid and points n of the unit sphere. This enables one to transform functions defined on the surface of the ellipsoid into functions defined on the surface of the unit sphere. In particular, let $\sigma(n)$ represent the stress on the surface of the ellipsoid.

For the general case of an ellipsoidal inclusion in an anisotropic medium, it was shown in [3] that the surface stress $\sigma(n)$ admits the representation

$$\sigma(n) = S(n)\mathscr{D}^{-1}\sigma^\circ, \tag{4.3}$$

where σ° denotes the stress at infinity, and $S(n)$ is a fourth-order tensor function of n and the elastic constants. The constant fourth-order tensor $\mathscr{D}$ is given by

$$\mathscr{D} = \langle S(n)\rangle = \frac{\det a}{4\pi}\int S(n)\rho^3(n)\,dn, \tag{4.4}$$

which is the mean value over the unit sphere with respect to the weight function

$$\rho(n) = (na^2n)^{-1/2}. \tag{4.5}$$

For the case of an ellipsoidal void in an isotropic medium, $S(n)$ and $\mathscr{D}$ are given explicitly in [3], $\mathscr{D}$ being expressed in terms of elliptic integrals. These results enable one to show that

$$\sigma_{ij}(n) = \tau(\delta_{ij} - n_i n_j) \tag{4.6}$$

if and only if σ° is given by

$$\sigma^\circ_{ij} = \frac{\lambda + 2\mu}{2\mu(3\lambda + 2\mu)}\tau\mathscr{D}_{ijkk}. \tag{4.7}$$

Tedious, but routine calculations lead to the conclusion that the principal axes of σ° coincide with those of the ellipsoid, and the principal stresses σ°_1, σ°_2, σ°_3 are expressed as

$$\sigma^\circ_p = (1 - I_p), \tag{4.8}$$

$$I_p = \frac{\det a}{2}\int_0^\infty \frac{d\xi}{(a_p{}^2 + \xi)\sqrt{((a_1{}^2 + \xi)(a_2{}^2 + \xi)(a_3{}^2 + \xi))}}. \tag{4.9}$$

It is easy to show by direct computation that the integrals I_p satisfy

$$I_1 + I_2 + I_3 = 1. \tag{4.10}$$

Accordingly, we may express σ°_p as

$$\sigma^\circ_p = \tau\sum_{k\neq p} I_k. \tag{4.11}$$

It is clear from these expressions that ellipsoids for which (4.6) hold do not exist for arbitrary σ°_p. In particular, since the integrals I_p are non-negative, the three quantities σ°_1, σ°_2, σ°_3 must have the same sign. Furthermore, a calculation based upon (4.11) leads to

$$(\operatorname{tr}\sigma)^2 - 2|\sigma^\circ|^2 = (2\tau)^2(I_1I_2 + I_1I_3 + I_2I_3), \tag{4.12}$$

so that σ° must conform to the inequality

$$|\operatorname{tr}\sigma^\circ| \geq \sqrt{2}|\sigma^\circ|. \tag{4.13}$$

This inequality ensures that the maximum of $|\sigma|$ occurs at the surface of the void, and that

$$\|\sigma^*\| = \frac{1}{\sqrt{2}}|\operatorname{tr}\sigma^\circ|. \tag{4.14}$$

Our approach here has been to start with an ellipsoid and deduce stresses at infinity for which the condition (4.6) holds. A preferable way would be to start with σ° and deduce the corresponding shape or shapes for which (4.6) is satisfied. It would therefore be of interest to investigate the invertibility of the mapping $(a_1, a_2, a_3) \rightarrow (\sigma_1^\circ, \sigma_2^\circ, \sigma_3^\circ)$ defined by (4.8).

REFERENCES

1. L. T. Wheeler, On the role of constant stress surfaces in the problem of minimizing elastic stress concentration. *Int. J. Solids Structures* 12, 779–789 (1976).
2. L. T. Wheeler, On optimum profiles for the minimization of elastic stress concentration. *ZAMM* 58, T235–T236 (1978).
3. I. A. Kunin and E. G. Sosnina, Stress concentration on an ellipsoidal inhomogeneity in an anisotropic elastic medium. *Prikl. Mat. Mekh.* 37, 306–315 (1973).
4. M. E. Gurtin and E. Sternberg, Theorems in linear elastostatics for exterior domains. *Arch. Rational Mech. Anal.* 8, 99–119 (1961).
5. M. E. Gurtin, *The Linear Theory of Elasticity*. Handbuch der Physik, Volume VIa/2, Springer-Verlag (1972).
6. O. D. Kellogg, *Foundations of Potential Theory*. Springer-Verlag (1929), Dover Publications reissue (1953).